# SYDNEY TECHNICAL HIGH SCHOOL

## HIGHER SCHOOL CERTIFICATE
## ASSESSMENT TASK 3

## JUNE 2013

# Mathematics

**General Instuctions**

- Working time - 70 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in questions 6 to 13
- Start each question on a new page
- A table of standard integrals is provided at the back of the paper

**Total marks - 55**

**Section 1 - 5 marks**

Attempt Questions 1 – 5.
Allow about 7 minutes for this section.

**Section 2 - 50 marks**

Attempt Questions 6 – 11.
Allow about 63 minutes for this section.

**Name :** ____________________

**Teacher :** ____________________

## Section 1

5 marks
Attempt Questions 1 – 5
Allow about 7 minutes for this section

Use the multiple-choice answer sheet in your answer booklet for Questions 1 – 5.
Do not remove the multiple-choice answer sheet from your answer booklet.

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**1.** What is the period of the function $y = 5 - 3\cos 2x$ ?

A) 3

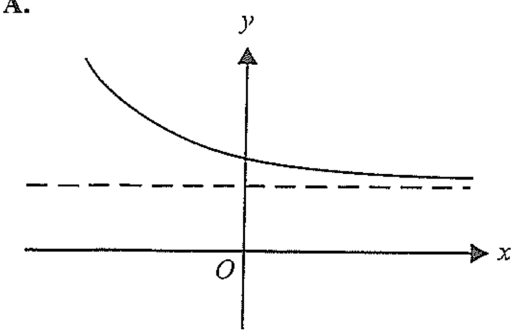
B) 5

C) $4\pi$

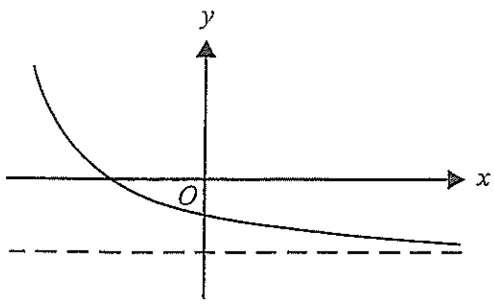
D) $\pi$

**2.** If $k$ is a negative real number and $P$ is a positive real number, which one of the following is most likely to be the graph of the function with equation $y = e^{kx} + P$ ?

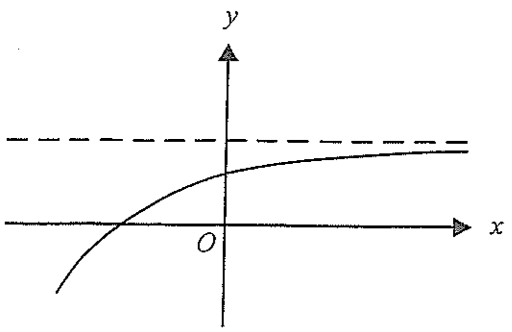
**A.**



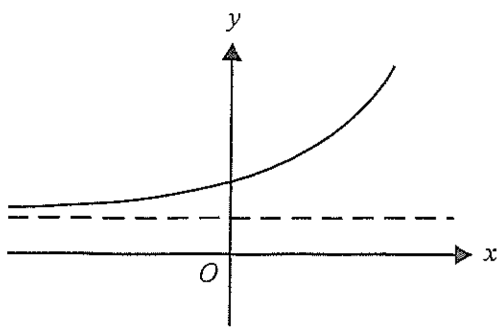

**B.**




**C.**




**D.**



3. If $\int_0^a sec^2 2x \; dx = \frac{1}{2}$ , then $a$ is equal to

A. $\frac{\pi}{4}$

B. $\frac{\pi}{8}$

C. $\frac{\pi}{12}$

D. $\frac{\pi}{2}$

4. If $g(t) = e^{-t} - 1$ then $g'(0)$ equals

A. $-e$

B. $-2$

C. $-1$

D. $0$

5. If $\int_1^3 f(x) \, dx = 5$ then $\int_1^3 (2f(x) - 3) \, dx$ is equal to

A. 4

B. 5

C. 7

D. 10

## Section 2

50 marks
Attempt Questions 6 – 10
Allow about 63 minutes for this section
Start each question on a new page

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**Question 6** (10 marks)

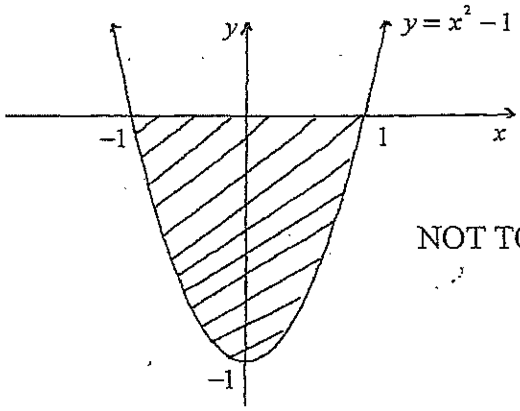
a) Evaluate $e^{-2.8}$ giving your answer correct to 2 significant figures. 2

b) Find the exact value of $\cos\frac{5\pi}{6}$ . 1

c) Sketch the graph of $y = 1 - \cos x$ for $0 \leq x \leq 2\pi$ . 2

d) Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x \; dx$ 2

e) The area bounded by $y = x^2 - 1$ and the $x$ - axis is rotated about the $y$ - axis . Find the volume of the solid of revolution formed. 3




NOT TO SCALE

**Question 7** (10 marks) (Start a new page)

a) Solve $\sqrt{3}\tan\theta - 1 = 0$ for $0 \leq \theta \leq 2\pi$. 2

b) Differentiate with respect to $x$.

i) $\tan 3x + \sin x$ 2

ii) $(x+2)e^{2x}$ 2

c) Consider the function $f(x) = \frac{x^2}{x+4}$

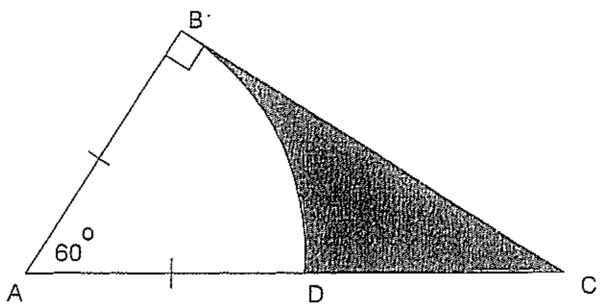
i) Use the trapezoidal rule with 4 function values to approximate $\int_1^7 f(x)\,dx$, giving your answer correct to 1 decimal place. 3

ii) For all values of $x$, between 1 and 7 , $f(x) > 0$ , $f'(x) > 0$ and $f''(x) > 0$ .

Use this information to decide whether the approximation found in part i) is an over-estimate or an under-estimate of the true value of the integral. Give a brief reason. 1

**Question 8** (10 marks) (Start a new page)

a)




In the diagram above, angle B = 90°, angle A = 60° and $AB = AD = 10\ m$.

$BD$ is an arc of the circle with centre $A$.

i) Calculate the exact length of the arc BD . 1

ii) Calculate the shaded area in exact form. 3

b) The area bounded by $y = x^2$ and $y = 4$ is rotated about the $x - axis$ to form a solid. Find the volume of the solid. 3

c) Find the equation of the tangent to $y = e^{4x} + x$ at the point where $x = 0$. 3

**Question 9** (10 marks) (Start a new page)

a) i) Draw a neat sketch of the curve 2

$y = 3\sin\frac{x}{2}$ for $-2\pi \le x \le 2\pi$ ,

showing clearly all the important features.

ii) Draw on your diagram a line, clearly labelled, which can be used to solve the following equation : 1

$3\sin\frac{x}{2} - x - 1 = 0$

iii) Determine the number of solutions the equation 1

$3\sin\frac{x}{2} - x - 1 = 0$ has over the domain $-2\pi \le x \le 2\pi$ .

b) i) Show that $\frac{d^2}{dx^2}(e^x \sin x) = 2e^x \cos x$ 2

ii) Hence find $\int e^x \cos x \, dx$ 2

c) For what values of $k$ does $y = 3e^{kx}$ satisfy the equation $\frac{d^2y}{dx^2} - 9y = 0$ ? 2

**Question 10** (10 marks) (Start a new page)

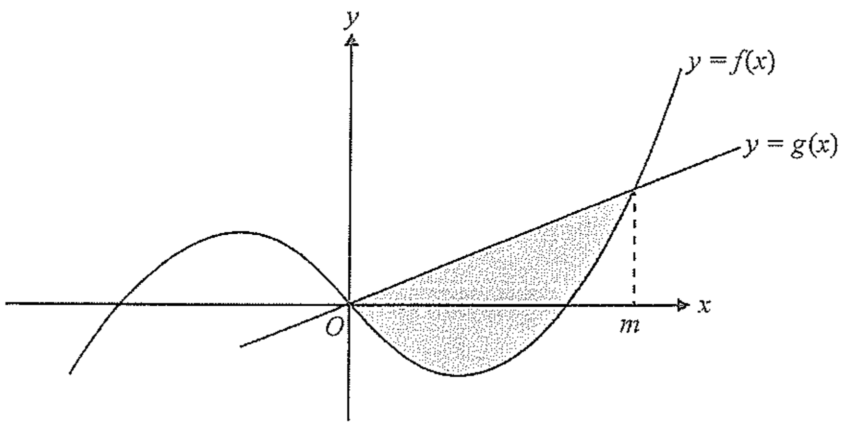
a) A function $f(x)$ is defined by $f(x) = \frac{e^{-x}}{x}$.

i) Differentiate $f(x)$ with respect to $x$. 2

ii) Find the coordinates of any stationary points of the graph of $y = f(x)$, and determine their nature. 2

iii) Sketch the graph of $y = f(x)$, showing all important features. (Not inflexion points) 2

b)




Parts of the graphs of the functions $f(x) = x^3 - ax$ , $a > 0$ and

$$g(x) = ax , a > 0$$

are shown in the diagram above.

The graphs intersect when $x = 0$ and when $x = m$. $(m \neq 0)$

i) Show that $m^2 = 2a$. 2

ii) If the area of the shaded region is 64 square units, find the value of $a$ and $m$. 2

**End of Paper**

## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

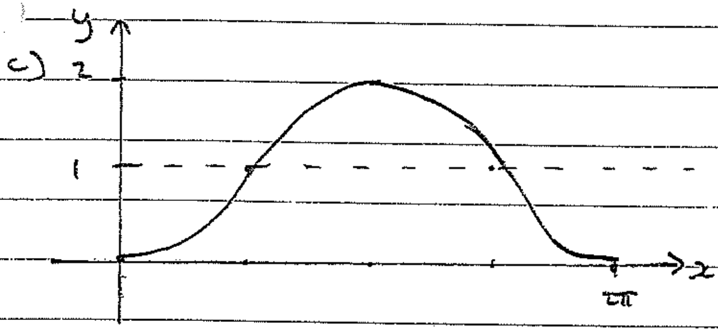
# SOLUTIONS

1. D
2. A
3. B
4. C
5. A

6. a) $0.061$

b) $-\frac{\sqrt{3}}{2}$

c)




d) $\int_0^{\frac{\pi}{6}} \sin 2x \, dx$

$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$

$= \left(-\frac{1}{2}\cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0\right)$

$= -\frac{1}{4} + \frac{1}{2}$

$= \frac{1}{4}$

e) $V = \pi \int_{-1}^{0} x^2 \, dy$

$= \pi \int_{-1}^{0} y + 1 \, dy$

$= \pi \left[\frac{1}{2}y^2 + y\right]_{-1}^{0}$

$= \pi \left( (0) - \left(\frac{1}{2} - 1\right)\right)$

$= \frac{\pi}{2}$ cu. units

7. a) $\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$

b) i) $3\sec^2 3x - \sin x$

ii) $e^{2x} + 2(x+2)e^{2x}$

$= e^{x}(2x+5)$

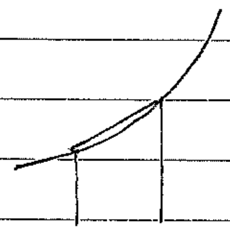
c) i)

| $x$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $f(x)$ | $\frac{1}{5}$ | $\frac{9}{7}$ | $\frac{25}{9}$ | $\frac{49}{11}$ |

$\therefore \int_1^7 f(x)\,dx \doteqdot \frac{2}{2}\left[\frac{1}{5} + \frac{49}{11} + 2\left(\frac{9}{7} + \frac{25}{9}\right)\right]$

$= 12.8$

ii) curve is increasing and concave up



$\therefore$ each trapezium larger than actual area

$\therefore$ over-estimate

8.

a. $60° = \frac{\pi}{3}$

i) $l = r\theta$

$= 10 \times \frac{\pi}{3}$

$= \frac{10\pi}{3}$ m

ii) $\tan 60° = \frac{BC}{10}$

$BC = 10\sqrt{3}$

Area = $\Delta ABC$ − sector

$= \frac{1}{2} \times 10\sqrt{3} \times 10 - \frac{1}{2} \times 10^2 \times \frac{\pi}{3}$

$= 50\sqrt{3} - \frac{50\pi}{3}$ m²

b. $V = 2\pi \int_0^2 4^2 - (x^2)^2 \, dx$

$= 2\pi \left[ 16x - \frac{1}{5}x^5 \right]_0^2$

$= 2\pi \left[ 32 - \frac{32}{5} \right]$

$= \frac{256\pi}{5}$ cu units

c. $y = e^{4x} + x$

$y' = 4e^{4x} + 1$

when $x = 0$

$y' = 4e^0 + 1 = 5$ $\quad y = e^0 + 0 = 1$

$\therefore m = 5$ passing thro' $(0,1)$

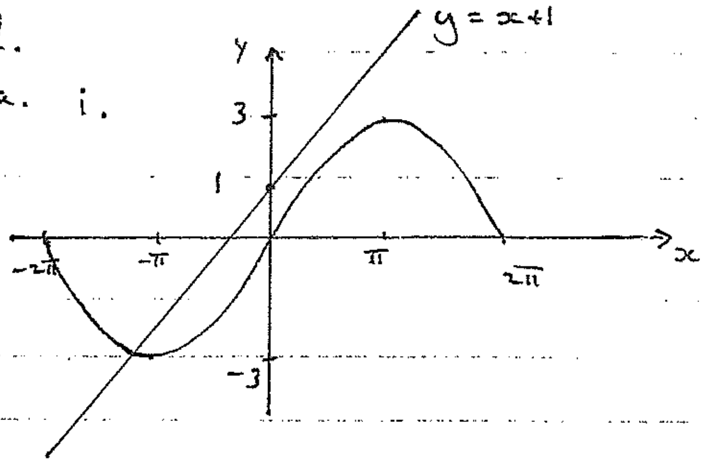
$y - y_1 = m(x - x_1)$

$y - 1 = 5(x - 0)$

$5x - y + 1 = 0$

9.

a. i.




ii. on diagram $y = x + 1$

iii. 1 solution

b. i. $y = e^x \sin x$

$\therefore \frac{dy}{dx} = e^x \sin x + e^x \cos x$

$\therefore \frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$

$= 2e^x \cos x$

ii. $\int e^x \cos x \, dx$

$= \frac{1}{2}(e^x \sin x + e^x \cos x) + c$

c. $y = 3e^{kx}$

$y' = 3ke^{kx}$

$y'' = 3k^2e^{kx}$

$\therefore y'' - 9y = 0$

$3k^2e^{kx} - 27e^{kx} = 0$

$3e^{kx}(k^2 - 9) = 0$

$\therefore k = \pm 3$

10.

a. i. $f(x) = \dfrac{e^{-x}}{x}$

$f'(x) = \dfrac{-xe^{-x} - e^{-x}}{x^2}$

$= \dfrac{-e^{-x}(x+1)}{x^2}$

ii. st. pts when $f'(x) = 0$

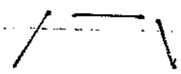
$\therefore x+1 = 0 \qquad (e^{-x} \neq 0)$

$x = -1$

$\therefore y = -e$

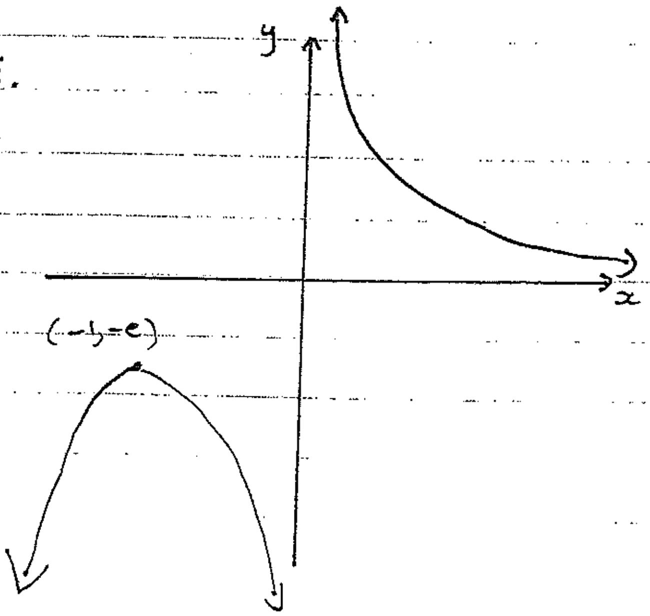
test

| $x$ | $-2$ | $-1$ | $-\frac{1}{2}$ |
|---|---|---|---|
| $y'$ | +ve | 0 | −ve |



$\therefore$ max at $(-1, -e)$

iii.




b. i. $x^3 - ax = ax$

$x^3 - 2ax = 0$

$x = m$ satisfies this equation

$\therefore m^3 - 2am = 0$

$m(m^2 - 2a) = 0$

$m^2 - 2a = 0 \qquad (m \neq 0)$

$\therefore m^2 = 2a$

ii. $\displaystyle\int_0^m ax - (x^3 - ax)\,dx = 64$

$\displaystyle\int_0^m 2ax - x^3\,dx = 64$

$\left[ax^2 - \frac{1}{4}x^4\right]_0^m = 64$

$(am^2 - \frac{1}{4}m^4) - (0) = 64$

but $m^2 = 2a$

$a(2a) - \frac{1}{4}(2a)^2 = 64$

$a^2 = 64$

$a = 8 \qquad (a > 0)$

$\therefore m = 4$.